Journal of Statistical Physics, Vol. 127, No. 3, May 2007 (© 2007)
DOI: 10.1007/s10955-006-9270-0

Finite Time Approach to Equilibrium in a Fractional Brownian Velocity Field

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Received October 10, 2005; accepted December 10, 2006
Published Online: February 23, 2007

We consider the solution of the equation $\dot{r}(t) = W(r(t)), r(0) = r_0 > 0$ where $W(\cdot)$ is a fractional Brownian motion (f.B.m.) with the Hurst exponent $\alpha \in (0, 1)$. We show that for almost all realizations of $W(\cdot)$ the trajectory reaches in finite time the nearest equilibrium point (i.e. zero of the f.B.m.) either to the right or to the left of r_0, depending on whether $W(r_0)$ is positive or not. After reaching the equilibrium the trajectory stays in it forever. The problem is motivated by studying the separation between two particles in a Gaussian velocity field which satisfies a local self-similarity hypothesis. In contrast to the case when the forcing term is a Brownian motion (then an analogous statement is a consequence of the Markov property of the process) we show our result using as the principal tools the properties of time reversibility of the law of the f.B.m., see Lemma 2.4 below, and the small ball estimate of Molchan, *Commun. Math. Phys.* **205** (1999) 97–111.

KEY WORDS: Passive tracer; fractional Brownian motion; two point separation function.

1. INTRODUCTION

In the passive tracer model, which is quite frequently used in statistical fluid mechanics to describe motion of particles in a turbulent flow (see e.g. Refs. 5, 7, or 10), the trajectory of a particle is given as a solution of the ordinary differential equation

$$\frac{d\mathbf{x}(t)}{dt} = V(t, \mathbf{x}(t)), \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{1.1}$$

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where the right hand side $V : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is a random field defined over a certain probability space $(\Omega, \mathcal{W}, \mathbb{P})$. By $\mathbb{E}$ we shall denote the expectation corresponding to the probability measure $\mathbb{P}$. The principal task is to describe the statistical properties of the flow of particles, given the statistics of the velocity field $V(t, \mathbf{x})$. Consider two particles whose trajectories $\mathbf{x}_i(t), i = 1, 2$ solve (1.1) with the initial conditions $\mathbf{x}_i(0) = \mathbf{x}_i^{(0)}, i = 1, 2$. We wish to analyze the separation between particles defined as $\mathbf{r}(t) := \mathbf{x}_2(t) - \mathbf{x}_1(t)$. It can easily be deduced that the separation vector satisfies the following equation

$$\frac{d\mathbf{r}(t)}{dt} = V(t, \mathbf{r}(t) + \mathbf{x}_1(t)) - V(t, \mathbf{x}_1(t)), \quad \mathbf{r}(0) = \mathbf{x}_2^{(0)} - \mathbf{x}_1^{(0)}. \tag{1.2}$$

Introducing $V^{qL}(t, \mathbf{x}) := V(t, \mathbf{x} + \mathbf{x}_1(t))$, the so-called *quasi-Lagrangian* velocity field, we can rewrite (1.2) in the form

$$\frac{d\mathbf{r}(t)}{dt} = V^{qL}(t, \mathbf{r}(t)) - V^{qL}(t, \mathbf{0}), \quad \mathbf{r}(0) = \mathbf{x}_2^{(0)} - \mathbf{x}_1^{(0)}. \tag{1.3}$$

One can inquire then about statistical properties of the separation vector assuming that the statistics of the quasi-Lagrangian field are known. Various results concerning the behavior of $\mathbf{r}(t)$ are presented in the extensive paper.[2]

Assume now that $d = 1$. One-dimensional models are often used e.g. to describe transport in isotropic flows. Suppose furthermore that the quasi-Lagrangian velocity is a time independent, zero mean, Gaussian random field that satisfies *the local self-similarity hypothesis*, with a fixed Hurst exponent α. Its covariance function can be described by

$$R(x) := \mathbb{E}[V^{qL}(x)V^{qL}(0)] = E_0 \int_{\mathbb{R}} \frac{e^{ikx}}{|k|^{1+2\alpha}} \mathbf{1}_{[L_0^{-1}, \ell_d^{-1}]}(|k|)\, dk. \tag{1.4}$$

Here L_0, ℓ_d stand for the energy-containing and dissipative scales, respectively, while E_0 is a parameter that can be used to control the magnitude of the energy flux. The scale separation satisfies $L_0/\ell_d \to +\infty$, as the Reynolds number of the flow $\mathrm{Re} \to +\infty$ (in fact one can argue that for $\alpha = 1/3$ the ratio L_0/ℓ_d is at least of the order of magnitude $\mathrm{Re}^{3/4}$, see e.g. (7.18) of Ref. 5). The corresponding pair separation function is given by $r_{\ell_d, L_0}(t)$—the solution of equation

$$\frac{dr_{\ell_d, L_0}(t)}{dt} = W_{\ell_d, L_0}(r_{\ell_d, L_0}(t)), \quad r_{\ell_d, L_0}(0) = r_0. \tag{1.5}$$

Here $W_{\ell_d, L_0}(x) := [V^{qL}(x) - V^{qL}(0)]$. By verifying the convergence of the respective covariance functions it can be shown that the laws of $W_{\ell_d, L_0}(x)$ converge weakly, as $\ell_d \ll 1 \ll L_0$, to the law of a fractional Brownian motion (f.B.m.) $W(x)$ with the Hurst exponent α and variance c_α. The above means that $W(x)$ is a Gaussian process which satisfies

$$W(0) = 0, \quad \mathbb{E}(W(x) - W(y))^2 = c_\alpha |x - y|^{2\alpha}$$

for all $x, y > 0$. Here

$$c_\alpha = 4E_0 \int_0^{+\infty} \frac{\sin^2(k/2)}{k^{2\alpha+1}} dk.$$

The corresponding limiting pair separation process $r(t)$ should satisfy therefore (see Theorem 1.2 below for a rigorous derivation) the equation

$$\frac{dr(t)}{dt} = W(r(t)), \quad r(0) = r_0 > 0, \tag{1.6}$$

were $W(x)$ denotes the f.B.m. described above. The Eq. (1.6) is our main object of interest in this paper. Following the above motivation, we study it only for $x > 0$, but the results could be extended to the whole real line with an appropriate definition of $W(x)$ for x negative.

What we set out to prove in this article is the almost sure (a.s.) uniqueness result for the solutions of (1.6). Informally speaking the behavior of solutions can be described as follows. Let $r_0 > 0$ be fixed. Suppose also that $W(r_0) > 0$. We define then

$$\rho_{r_0} := \inf\,[r > r_0 : W(r) = 0]\,. \tag{1.7}$$

When, on the other hand $W(r_0) < 0$ we set

$$\rho_{r_0} := \sup\,[r < r_0 : W(r) = 0]\,. \tag{1.8}$$

In case when $W(r_0) = 0$, which is a probability zero event, we define by convention $\rho_{r_0} := r_0$. If $r_0 > 0$ and $W(r_0) \neq 0$ then, as we show in Proposition 2.2, one has $0 \leq \rho_{r_0} < +\infty$ a.s. The solution of (1.6) is uniquely determined by the equation

$$\int_{r_0}^{r(t)} \frac{dr}{W(r)} = t, \tag{1.9}$$

so long as $W(r(t)) \neq 0$. In writing formula (1.9) we have adopted the usual convention that $\int_b^a f(x)dx = -\int_a^b f(x)dx$, when $a < b$. According to Proposition 2.5 below the trajectory $r(t)$ must reach in finite time the point ρ_{r_0} a.s. After that time, thanks to Proposition 2.7, the solution of (1.6) must stay at ρ_{r_0} for all the remaining time.

To formulate the result rigorously let us introduce a strictly monotone function

$$F(r) := \int_{r_0}^{r} \frac{d\rho}{W(\rho)}, \tag{1.10}$$

It is well defined, by means of formula (1.10), for all those ω for which $W(r_0; \omega) \neq 0$ and r belonging to an oriented interval $[r_0, \rho_{r_0})$. In case when $W(r_0; \omega) = 0$ we shall simply set $F(r; \omega) \equiv 0$.

Theorem 1.1. *For any* $r_0 > 0$ *we have*

$$\tau_0 := F(\rho_{r_0}-) < +\infty, \quad \mathbb{P} - a.s. \tag{1.11}$$

The Eq. (1.6) has a unique solution, given by the formula

$$r(t) = \begin{cases} F^{-1}(t), & t \in [0, \tau_0), \\ \rho_{r_0}, & t \in [\tau_0, +\infty), \end{cases}$$

when $W(r_0) \neq 0$.

In the special case when $\alpha = 1/2$ and $W(\cdot)$ is a Brownian motion this result can be proven very simply (for example using Itô stochastic calculus), but all such simple proofs known to the authors rely, more or less explicitly, on the Markov property of the process. Since the fractional Brownian motion no longer has the Markov property we have to use a different approach. Our proof rests on a time reversibility property of the process, see Lemma 2.4 and Theorem 2.3 and on the small ball estimates of Ref. 9, see (2.17) below.

As an application of Theorem 1.1 we prove, see Sec. 3 below, the following result that makes the formal passage from the separation process $r_L(t)$ to the solution of (1.6) rigorous.

Theorem 1.2. *The laws of processes* $r_{\ell_d, L_0}(t), t \geq 0$, *given by (1.5), converge weakly on the space* $C[0, +\infty)$, *as* $\ell_d \ll 1 \ll L_0$, *to the law of the solution of (1.6).*

2. SOME PROPERTIES OF THE FRACTIONAL BROWNIAN MOTION

Suppose that $u \in \mathbb{R}$. Define

$$\sigma_u := \inf[x > 0 : W(x) = -u]. \tag{2.1}$$

Lemma 2.1. *Let* $A := [\omega : \sigma_u(\omega) < +\infty$ *for all* $u \in \mathbb{R}]$. *Then* $\mathbb{P}[A] = 1$.

Proof: Let $u \in \mathbb{R}$ and $A_u := [\omega : \sigma_u < +\infty]$. The conclusion of the lemma shall follow from the fact that

$$\mathbb{P}[A_u] = 1 \quad \text{for all } u \in \mathbb{R}. \tag{2.2}$$

Indeed, (2.2) would certainly imply that

$$\mathbb{P}[\omega : \sigma_u < +\infty \quad \text{for all } u \in \mathbb{Q}] = 1. \tag{2.3}$$

Since $\sigma_u \leq \sigma_v$ when either $0 \leq u \leq v$, or $v \leq u \leq 0$, and the paths of the process W are continuous, Eq. (2.3) implies the conclusion of the lemma. Suppose that $u \neq 0$ is such that $\mathbb{P}[A_u^c] = c > 0$. With no loss of generality we may and shall assume that $u > 0$. Thanks to α-self similarity of the fractional Brownian motion, i.e. the fact that the laws of $W(x)$, $x \geq 0$ and $a^{\alpha} W(x/a)$, $x \geq 0$ are identical for all $a > 0$, we conclude that also $\mathbb{P}[A^c_{a^{\alpha}u}] = c > 0$ for all $a > 0$. Hence,

$$\mathbb{P}\big[M_{[0,1]} = 0\big] \geq c > 0, \tag{2.4}$$

where

$$M_{[x,y]} := \min_{r \in [x,y]} W(r). \tag{2.5}$$

It is known, see e.g. Proposition 4 of Ref. 8, that $M_{[0,1]}$ has an absolutely continuous law w.r.t. the Lebesgue measure, which clearly contradicts (2.4). This ends the proof of (2.2) and, in consequence, concludes the proof of the lemma. □

Using the above lemma we obtain the following.

Proposition 2.2. *Suppose that $r_0 > 0$. Then, for ρ_{r_0} defined in (1.7) we have*

$$\rho_{r_0} < +\infty, \quad \mathbb{P} - a.s. \tag{2.6}$$

Proof: The case $W(r_0) < 0$ is obvious. We only need to be concerned with the case $W(r_0) > 0$. Consider then the f.B.m. given by $\widetilde{W}(r) := W(r + r_0) - W(r_0)$, $r \geq 0$. We emphasize here that, in contrast with the standard Brownian case, this process is not independent of $W(r)$, $r \in [0, r_0]$. Note that $\rho_{r_0} = \tilde{\sigma}_{W(r_0)} - r_0$, where $\tilde{\sigma}_u$ is defined as in (2.1) with the f.B.m. $W(r)$, $r \geq 0$ replaced by $\widetilde{W}(r)$, $r \geq 0$. The result immediately follows then from Lemma 2.1. □

The main result of this section is the following.

Theorem 2.3. *We have*

$$\int_{-v}^{v} \mathbb{E}\left[\int_0^{\sigma_u} \frac{dr}{|u + W(r)|} \mathbf{1}_{[\sigma_u < X]}\right] du < +\infty \tag{2.7}$$

for any $v, X > 0$, where $\mathbf{1}_{[\sigma_u < X]}$ denotes the indicator function corresponding to the event $[\sigma_u < X]$.

Proof: Suppose that $\Omega = C[0, +\infty)$, $\mathcal{M}$ is its Borel σ-algebra, $\mathbb{W}$ is the law of the f.B.m. and $W(x; \omega) := \omega(x)$ is the coordinate map. On the Borel σ-algebra $\mathcal{B}$ of subsets of $\mathbb{R} \times \Omega$ we introduce a measure of infinite total mass $\mu := \lambda \otimes \mathbb{W}$, where λ is the standard one-dimensional Lebesgue measure. Likewise, for a given $X > 0$ we denote $\Omega_X = C[0, X]$ and write $\mathcal{M}_X$ for the corresponding Borel σ-algebra. Let $\pi_X : \mathbb{R} \times \Omega \to \mathbb{R} \times \Omega_X$ be given by $\pi_X(x, \omega) := (x, \omega_{|[0,X]})$, where $\omega_{|[0,X]}$ is the restriction of ω to $[0, X]$. Then $\mu_X := \mu \pi_X^{-1}$ defines a measure on the Borel σ-algebra of subsets of $\mathbb{R} \times \Omega_X$.

To prove the assertion of the theorem it suffices to show that for any $u > 0$.

$$\int \left[\int_0^X \frac{\mathbf{1}_{[0,\sigma_u]}(r)dr}{u + W(r)} \right] \mathbf{1}_{[\sigma_u < X]} \mathbf{1}_{[0,v]}(u) \mu_X(du, d\omega) < +\infty \tag{2.8}$$

for any $v, X > 0$. The statement with $\mathbf{1}_{[0,v]}$ replaced by $\mathbf{1}_{[-v,v]}$ follows from the symmetry of the law of the f.B.m. under the spatial reflection.

The integral in (2.8) is clearly bounded from above by

$$\int \left[\int_0^X \frac{\mathbf{1}_{[0,\sigma_u]}(r)dr}{u + W(r)} \right] \quad \mathbf{1}_{[0,v]}(u) \mu_X(du, d\omega). \tag{2.9}$$

For any $r \le X$ we let

$$\mathcal{W}(w, r, v) := \mu_X[(u, \omega) : 0 \le u + W(r; \omega) \le w, r \le \sigma_u(\omega), 0 \le u \le v]. \tag{2.10}$$

To prove that the expression in (2.9) is bounded it suffices only to show that

$$\int_0^X dr \int_0^{+\infty} \frac{1}{w} \mathcal{W}(dw, r, v) < +\infty, \tag{2.11}$$

Indeed, for any $g : [0, +\infty) \to \mathbb{R}$ that is smooth and $\lim_{w\to\infty} g(w) = 0$ one can easily prove, using integration by parts formula, that

$$\begin{aligned} &\int \left[\int_0^X g(u + W(r)) \mathbf{1}_{[0,\sigma_u]}(r) dr \right] \quad \mathbf{1}_{[0,v]}(u) \mu_X(du, d\omega) \\ &= \int_0^X dr \int_0^{+\infty} g(w) \mathcal{W}(dw, r, v). \end{aligned} \tag{2.12}$$

Then choosing an increasing sequence of smooth, positive functions $g_n(w)$ tending to $1/w$ on $[0, +\infty)$, as $n \to +\infty$, and using Beppo Levi monotone convergence theorem one can conclude that (2.11) implies finiteness of the expression in (2.9). Moreover, the expressions appearing in (2.9) and (2.11) must be equal.

The crucial property of measure μ_X that we are going to apply in proving (2.11) is its invariance under the time reversal transformation. It is expressed by the following.

Lemma 2.4. *Suppose that $X > 0$ and $g : \Omega_X \to \mathbb{R}$ is a function integrable with respect to the measure μ. Let also $R_X : \Omega_X \to \Omega_X$ be given by $R_X(\omega)(x) := \omega(X - x)$, $x \in [0, X]$. Then,*

$$\int g(u + R_X(\omega)) \mu_X(du, d\omega) = \int g(u + \omega) \mu_X(du, d\omega). \tag{2.13}$$

Here, $(u + \omega)(x) := u + \omega(x)$, $x \in [0, X]$, $u \in \mathbb{R}$.

We postpone for a moment the proof of this result in order to finish proving the theorem. With the above lemma we can write for any $r \le X$

$$\begin{aligned}
\mathcal{W}(w, r, v) &= \mu_X\Big[(u, \omega) : 0 \le u + W(r; \omega) \le w, \\
&\qquad \inf_{y\in(0,r)} (u + W(y; \omega)) > 0, 0 \le u \le v\Big] \\
&= \mu_r\Big[(u, \omega) : 0 \le u + W(r; \omega) \le w, \\
&\qquad \inf_{y\in(0,r)} (u + W(y; \omega)) > 0, 0 \le u \le v\Big] \\
&\overset{(2.13)}{=} \mu_r\Big[(u, \omega) : 0 \le u \le w, \inf_{y\in(0,r)} (u + W(r - y; \omega)) > 0, \\
&\qquad 0 \le u + W(r; \omega) \le v\Big] \\
&= \int_0^w \mathbb{W}\Big[\omega : \inf_{y\in(0,r)} (u + W(y; \omega)) > 0, \\
&\qquad 0 \le u + W(r; \omega) \le v\Big] du. \qquad (2.14)
\end{aligned}$$

Using (2.14) we may estimate from above the expression appearing in (2.11) by

$$\int_0^X dr \int_0^{+\infty} \frac{dw}{w} \mathbb{W}\Big[\omega : \min_{y\in[0,r]} (w + W(u; \omega)) \ge 0, 0 \le w + W(r; \omega) \le v\Big]. \qquad (2.15)$$

The probability in (2.15) can be estimated from above by

$$\min\Big\{\mathbb{W}\Big[\omega : \min_{y\in[0,r]} (w + W(y; \omega)) \ge 0\Big], \mathbb{W}[W(r) \in [-w, v - w]]\Big\}$$

The first term under the minimum equals

$$\mathbb{W}\Big[\omega : \min_{y\in[0,r]} W(y; \omega) \ge -w\Big] = \mathbb{W}\Big[\omega : \max_{y\in[0,r]} W(y; \omega) \le w\Big]. \qquad (2.16)$$

According to Lemma 3, p. 105 of Ref. 9, for any $\epsilon > 0$ one can find a constant $C > 0$ such that the right hand side of (2.16) is estimated from above by

$$C w^{-1+\epsilon+1/\alpha} r^{-1-\epsilon+\alpha}, \quad \forall r, w > 0. \qquad (2.17)$$

We estimate the second term for $w \ge 2v \ge 0$ as follows

$$\begin{aligned}
\mathbb{W}[W(r) \in [-w, v - w]] &= \frac{1}{\sqrt{2\pi r^{2\alpha}}} \int_{-w}^{v-w} \exp\left\{-\frac{u^2}{2r^{2\alpha}}\right\} du \\
&\le \frac{1}{\sqrt{2\pi r^{2\alpha}}} \int_{-\infty}^{-w/2} \exp\left\{-\frac{u^2}{2r^{2\alpha}}\right\} du
\end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{w/(2r^{\alpha})}^{+\infty} \exp\left\{-\frac{u^2}{2}\right\} du$$

$$\leq \sqrt{\frac{2}{\pi}} \frac{r^{\alpha}}{w} \exp\left\{-\frac{w^2}{8r^{2\alpha}}\right\} \leq \sqrt{\frac{2}{\pi}} \frac{r^{\alpha}}{w}. \qquad (2.18)$$

The penultimate estimate follows from the elementary inequality $\int_x^{+\infty} e^{-y^2/2} dy \leq x^{-1} e^{-x^2/2}$ for $x > 0$ that can be directly verified (note that $\int_x^{+\infty} y e^{-y^2/2} dy = e^{-x^2/2}$, see also exercise 4, p. 212 of Ref. 3). Choosing $\epsilon \in (0, \alpha)$ and estimating the integrand in (2.15), using (2.17) for $w \in (0, 2v)$ and (2.18) for $w \geq 2v$, we conclude that the integral can be bounded from above by

$$C \int_0^X \int_0^{2v} w^{-2+\epsilon+1/\alpha} r^{-1-\epsilon+\alpha} dr\, dw + \sqrt{\frac{2}{\pi}} \int_0^X \int_{2v}^{+\infty} w^{-2} r^{\alpha} dr dw < +\infty, \qquad (2.19)$$

as claimed.

The Proof of Lemma 2.4. It suffices to show that for any $n \geq 0$, functions $g_0, \ldots, g_n \in C_0^{\infty}(\mathbb{R})$, arguments $0 = x_0 \leq x_1 < \cdots x_n \leq X$ we have

$$\int_{-\infty}^{+\infty} \mathbb{E}[g_0(v + W(x_0)) g_1(v + W(x_1)) \ldots g_n(v + W(x_n))]\, dv$$

$$= \int_{-\infty}^{+\infty} \mathbb{E}[g_0(v + W(X)) g_1(v + W(X - x_1)) \ldots g_n(v + W(X - x_n))] dv. \qquad (2.20)$$

We can rewrite the right hand side of the above equation using the Fourier representation for functions g_i:

$$\int \underset{\mathbb{R}^{n+1}}{\cdots} \int \hat{g}_0(k_0) \hat{g}_1(k_1) \ldots \hat{g}_n(k_n) \mathbb{E}\left[\exp\left(i \sum_{p=0}^{n} k_p W(X - x_p)\right)\right]$$

$$\times \left[\int \exp\left(i v \sum_{p=0}^{n} k_p\right) dv\right] dk_0 \ldots dk_n$$

$$= \int \underset{\mathbb{R}^{n}}{\cdots} \int \hat{g}_0\left(-\sum_{p=1}^{n} k_p\right) \hat{g}_1(k_1) \ldots \hat{g}_n(k_n)$$

$$\times \mathbb{E}\left[\exp\left(i \sum_{p=1}^{n} k_p [W(X - x_p) - W(X)]\right)\right] dk_1 \ldots dk_n. \qquad (2.21)$$

To obtain the last equality we used the fact that the last integral appearing on the left hand side of (2.21) equals $\delta(\sum_{p=0}^{n} k_p)$. Using the fact that the laws of $W(X-x) - W(X), x \in [0, X]$ and $W(x), x \in [0, X]$ coincide we conclude that the right hand side of (2.21) equals

$$\int \underset{\mathbb{R}^n}{\cdots} \int \hat{g}_0\left(-\sum_{p=1}^{n} k_p\right) \hat{g}_1(k_1)\ldots\hat{g}_n(k_n)\mathbb{E}\left[\exp\left(i\sum_{p=1}^{n} k_p W(x_p)\right)\right] dk_1\ldots dk_n$$

$$= \int \underset{\mathbb{R}^n}{\cdots} \int \hat{g}_0(k_0)\hat{g}_1(k_1)\ldots\hat{g}_n(k_n)$$

$$\times\, \mathbb{E}\left[\exp\left(i\sum_{p=1}^{n} k_p W(x_p)\right)\right]\left[\int \exp\left(iv\sum_{p=0}^{n} k_p\right) dv\right] dk_0\ldots dk_n. \quad (2.22)$$

The right hand side of Eq. (2.21) is equal to the left hand side of (2.20). □

As a direct consequence of Theorem 2.3 we have the following

Proposition 2.5. *Suppose that* $r_0 > 0$ *is such that* $W(r_0) \neq 0$. *Then,*

$$\int_{r_0}^{\rho_{r_0}} \frac{dr}{W(r)} < +\infty, \quad \mathbb{P} - a.s. \quad (2.23)$$

Proof: We show that both

$$\mathbf{1}_{[W(r_0)>0]} \int_{r_0}^{\rho_{r_0}} \frac{dr}{W(r)} < +\infty, \quad \mathbb{P} - a.s. \quad (2.24)$$

and

$$\mathbf{1}_{[W(r_0)<0]} \int_{r_0}^{\rho_{r_0}} \frac{dr}{W(r)} < +\infty, \quad \mathbb{P} - a.s. \quad (2.25)$$

Proof of (2.24). In view of Lemma 2.1 in order to prove (2.24) it suffices only to show that for any $X > 0$

$$\mathbf{1}_{[0<W(r_0),\rho_{r_0}<X]} \int_{r_0}^{\rho_{r_0}} \frac{dr}{W(r)} < +\infty, \quad \mathbb{P} - a.s. \quad (2.26)$$

Recall that $\widetilde{W}(r) := W(r + r_0) - W(r_0)$, $r \geq 0$ is a f.B.m. whose law coincides with that of $W(\cdot)$. For any $0 \leq x \leq y$ we denote by $\mathcal{F}_{x,y}$ the σ-algebra generated by $\widetilde{W}(r), r \in [x, y]$. By the orthogonal projection theorem the random variable $W(r_0)$ can be uniquely decomposed as a sum of zero-mean Gaussian random variables, see e.g. Theorem 10.1 p. 181 of Ref. 11, $M(\omega) + S(\omega)$, where $M(\omega)$ is $\mathcal{F}_{0,X-r_0}$—measurable and $S(\omega)$ is independent of $\mathcal{F}_{0,X-r_0}$. Let $\Sigma := \mathbb{E}S^2$. The conditional law of r.v. $W(r_0)$ w.r.t. $\mathcal{F}_{0,X-r_0}$ is therefore Gaussian with the mean $M(\omega)$ and variance Σ. □

Lemma 2.6. *We have* $\Sigma > 0$.

Proof: If $\Sigma = 0$, then $W(r_0)$ is measurable with respect to the σ-algebra generated by the random variables $W(x) - W(r_0)$, where $r_0 \le x \le X$. Hence $W(r_0;\omega) = f(\omega(\cdot) - W(r_0;\omega))$, $\mathbb{W}$-a.s., where $f : C[r_0, X] \to \mathbb{R}$ is Borel measurable. Let $T : C[r_0, X] \to C[r_0, X]$ be given by $T(\omega)(x) := \omega(x) + 1$. Since the interval $[r_0, X]$ does not contain zero, the measure $\mathbb{W} \circ T^{-1}$ is equivalent (i.e. mutually absolutely continuous) to $\mathbb{W}$. This is an immediate consequence of the Girsanov theorem for the fractional Brownian motion (Ref. 4, Theorem 4.9, p. 25), where we choose a drift $K_H(s,t) = u(s)$ so that $\int_0^{r_0} u(s) = 1$ and $u(s) \equiv 0$ for $s \in [r_0, X]$. Hence, in particular, we must have $W(r_0; T(\omega)) = f(T(\omega)(\cdot) - W(r_0; T(\omega)))$. Since $T(\omega)(\cdot) - W(r_0; T(\omega)) = \omega(\cdot) - W(r_0;\omega)$, this would imply $W(r_0) = W(r_0) + 1$ with probability one—a contradiction. □

Now, let us return to the proof of (2.24). In view of (2.26) it is enough to show that for any positive integer m and $X > 0$ we have

$$\mathbb{E}\left[\int_0^{\tilde{\sigma}_{W(r_0)}} \frac{dr}{W(r_0) + \widetilde{W}(r)}, \tilde{\sigma}_{W(r_0)} < X - r_0, |M(\omega)| \le n, 0 < W(r_0) \le m\right] < +\infty. \tag{2.27}$$

We recall that $\tilde{\sigma}_u$ is defined for the f.B.m. $\widetilde{W}(r)$ by (2.1). This is however an easy consequence of (2.7) since conditioning upon $\mathcal{F}_{0,X-r_0}$ we can rewrite the expectation on the left hand side of (2.27) as being equal to

$$\mathbb{E}\left[\int_0^m g_{M(\omega),\Sigma}(v)dv \int_0^{\tilde{\sigma}_v} \frac{dr}{v + \widetilde{W}(r)}, \tilde{\sigma}_v < X - r_0 |M(\omega)| \le n\right]$$
$$\le C_n \int_0^m \mathbb{E}\left[\left|\int_0^{\tilde{\sigma}_v} \frac{dr}{v + \widetilde{W}(r)}\right|, \tilde{\sigma}_v < X - r_0\right] dv < +\infty \tag{2.28}$$

by virtue of Theorem 2.3. Here, $g_{M,\Sigma}$ is the normal density with mean M and variance Σ. Thus, we conclude the proof of (2.24). □

Proof of (2.25). In this case we introduce the f.B.m. $\widehat{W}(r) := W(r_0 - r) - W(r_0)$, $r \in [0, r_0]$ and the corresponding stopping time $\hat{\sigma}_u$ defined for an arbitrary $u \in \mathbb{R}$ via (2.1) where $W(\cdot)$ is replaced by $\widehat{W}(\cdot)$. We shall also introduce $\hat{\mathcal{F}}_{x,y}$, $0 \le x \le y \le r_0$ the filtration of σ-algebras corresponding to $\widehat{W}(\cdot)$. Let k be a positive integer. The conditional law of r.v. $W(r_0)$ w.r.t. $\hat{\mathcal{F}}_{0,r_0-1/k}$ is therefore Gaussian with the mean $\hat{M}(\omega)$ and variance $\hat{S}(\omega)$. Arguing in the same manner as in the proof of Lemma 2.6 we convince ourselves that $\hat{\Sigma} := \mathbb{E}\hat{S}^2 > 0$. Note also that $[\hat{\sigma}_{W(r_0)} = r_0] = [\rho_{r_0} = 0]$ is contained in $[\max_{r\in[0,r_0]} W(r) = 0]$, which, as has been already observed, is a null event (see the remark below (2.5)). It is enough therefore to show that for any positive integers k, m and n we have

$$\mathbb{E}\left[\int_0^{\hat{\sigma}_{W(r_0)}} \frac{dr}{|W(r_0)+\widehat{W}(r)|}, \hat{\sigma}_{W(r_0)} \le r_0 - \frac{1}{k}, |\hat{M}(\omega)| \le n, -m \le W(r_0) < 0\right] < +\infty. \tag{2.29}$$

This integral can be however written as

$$\mathbb{E}\left[\int_{-m}^{0} g_{\hat{M}(\omega),\hat{\Sigma}}(v)dv \int_0^{\hat{\sigma}_v} \frac{dr}{|v+\widehat{W}(r)|}, \hat{\sigma}_v < r_0 - \frac{1}{k}, |\hat{M}(\omega)| \le n\right]$$
$$\le C_n \int_{-m}^{0} \mathbb{E}\left[\left|\int_0^{\hat{\sigma}_0} \frac{dr}{v+\widehat{W}(r)}\right|, \hat{\sigma}_v < r_0\right] dv < +\infty \tag{2.30}$$

by virtue of Theorem 2.3. This finishes the proof of (2.25) and thus concludes the proof of the proposition. □

Proposition 2.7. *Suppose that* $r_0 > 0$ *and* $\epsilon > 0$. *Let*

$$A(r_0,\varepsilon) := [\omega : W(r_0) > 0, W(\cdot) \text{ changes sign infinitely often in } [\rho_{r_0}, \rho_{r_0}+\varepsilon]]. \tag{2.31}$$

and

$$B(r_0,\varepsilon) := [\omega : W(r_0) < 0,\ W(\cdot) \text{ changes sign infinitely often in } (\rho_{r_0}-\varepsilon,\ \rho_{r_0})]. \tag{2.32}$$

Then $\mathbb{P}[A(r_0,\varepsilon)] = \mathbb{P}[B(r_0,\varepsilon)] = 1$.

Proof: Let

$$A^{(1)}(r_0,\varepsilon) := \{\omega : W(\cdot) \text{ changes sign finitely many times in } [\rho_{r_0}, \rho_{r_0}+\varepsilon]\}$$
$$\cap\{W(r_0) > 0\}$$

Note that

$$A^{(1)}(r_0,\varepsilon) \subset \bigcup_{r \ge r_0, r\in\mathbb{Q}} (D_r \cup E_r)$$

where

$$D_r := \left[\omega : \min_{u\in[r_0,r]} W(u) = 0\right]$$

and

$$E_r := \left[\omega : \max_{u\in[r_0,r]} W(u) = 0\right]$$

According to a general criterion for absolute continuity of extrema of Gaussian process, see Ref. 13, we have $\mathbb{P}[D_r] = \mathbb{P}[E_r] = 0$ for all $r \ge r_0$, $r \in \mathbb{Q}$ and we obtain that $\mathbb{P}[A^{(1)}(r_0, \varepsilon)] = 0$. Hence (2.31) follows. The proof of (2.32) is analogous. □

3. PROOFS OF THEOREMS 1.1 AND 1.2

Proof of Theorem 1.1. We consider only the case when $W(r_0) > 0$, the case $W(r_0) < 0$ can be concluded via the same argument. Note that the solution $r(t)$ of (1.6) is determined uniquely for $t \in [0, \tau_0)$ (cf. (1.11)) from (1.9). According to Proposition 2.5 we have $\tau_0 < +\infty$ a.s. and $r(\tau_0) = \rho_{r_0}$. In addition, since the fractional Brownian motion $W(\cdot)$ changes sign infinitely often to the right of ρ_{r_0} the only solution of (1.6) that starts at τ_0 from ρ_{r_0} must be a constant trajectory $r(t) \equiv \rho_{r_0}$, $t \geq \tau_0$. For, clearly, every such solution must satisfy $r(t) \geq \rho_{r_0}$ for all $t \geq \tau_0$ (since W is positive in an interval to the left of ρ_{r_0}). On the event $\rho_{r_0} < q$, where q is any real number, we have $r(t) < q$ for $t \geq \tau_0$, since by virtue of Proposition 2.7, there is an interval (a, b), contained in (ρ_{r_0}, q), on which $W(\cdot)$ is negative and hence no solution can ever reach b. Since this is true in particular for all rational q-s, the conclusion of the theorem follows. □

The Proof of Theorem 1.2. Since the topology of weak convergence of probability measures is metrizable it suffices to prove that for any sequence $L_n \to +\infty$, $\ell_m \to 0+$ the laws of $r_{\ell_m, L_n}(t)$, $t \geq 0$ converge weakly on $C[0 + \infty)$. As we have already mentioned in Sec. 1 the sequence of the laws of $W_{L_n}(x), x \geq 0$ converges weakly, on $C[0, +\infty]$, to the law of the f.B.m. $W(x), x \geq 0$. Using Skorokhod representation theorem (see e.g. Ref. 1 Theorem 1.6.7 p. 70), one can construct a probability space $(\tilde{\Omega}, \tilde{\mathcal{W}}, \tilde{\mathbb{P}})$, a sequence of processes $\tilde{W}_{L_n}(x), x \geq 0$ and a process $\tilde{W}(x), x \geq 0$ defined over the probability space such that the law of each $\tilde{W}_{L_n}(x), x \geq 0$ coincides with the law of $W_{L_n}(x), x \geq 0$ and the law of $\tilde{W}(x), x \geq 0$ agrees with the law of the f.B.m. $W(x), x \geq 0$. Furthermore, the processes $\tilde{W}_{L_n}(x), x \geq 0$ converge $\tilde{\mathbb{P}}$-a.s. as $n \to +\infty$, to $\tilde{W}(x), x \geq 0$ uniformly on compact intervals. Thanks to Theorem 1.1 the equation

$$\frac{d\tilde{r}(t; \tilde{\omega})}{dt} = \widetilde{W}(\tilde{r}(t; \tilde{\omega}); \tilde{\omega}), \quad t \geq 0, \quad \tilde{r}(0) = r_0 > 0$$

has a unique solution for $\tilde{\mathbb{P}}$-a.s. $\tilde{\omega}$. Using a standard argument from the theory of ordinary differential equations one can easily conclude that $\tilde{r}_{L_n}(t), t \geq 0$—the solutions of

$$\frac{d\tilde{r}_{L_n}(t)}{dt} = \widetilde{W}_{L_n}(\tilde{r}_{L_n}(t)), t \geq 0, \quad \tilde{r}_{L_n}(0) = r_0 > 0$$

converge $\mathbb{P}$-a.s. to $\tilde{r}(t), t \geq 0$ uniformly on compact intervals. Since the laws of $\tilde{r}_{L_n}(t)$, $t \geq 0$ and those of $r_{L_n}(t)$, $t \geq 0$ are identical we conclude from this the convergence of the laws of $r_{L_n}(t), t \geq 0$ as claimed in the assertion of the theorem.

ACKNOWLEDGEMENTS

The authors thank K. Gawędzki, A. Kupiainen and M. Vergassola for introducing them to the problem addressed in this paper and for the invitation to participate in the program on developed turbulence at the Schrödinger Institute in Vienna in June of the year 2002, where this work was started. Furthermore, the authors express their gratitude to an anonymous referee for careful reading of the manuscript and helpful comments. The research of T.K. was supported by KBN grant nr 2PO3A 031 23.

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